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Effect of the Noise Correlation Time on the Stationary Current of Brownian Particle in a Spatially Symmetric Periodic Potential

Wang-Quan He · Can-Jun Wang · Dong-Cheng Mei

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Abstract We investigate the noise-induced transport of Brownian particle in a deterministic spatial symmetrical periodic potential driven by colored cross correlation between a multiplicative white noise and an additive white noise. We derive the general formula of the stationary current. Based on numerical computation, we found that directed motion of the Brownian particles can be induced by the correlation time τ of cross correlation between the multiplicative noise and the additive noise and the current reversal and the direction of the current is controlled by the τ.

Keywords Stationary current · Noise correlation time · Periodic potential

1 Introduction

Transport phenomena play a crucial role in a large variety of processes in nature. In recent years, there has been an increasing interest in studying the fluctuation-induced transport of Brownian particle moving in a one-dimensional periodic potential, i.e. thermal ratchet (TR), subjected to different kinds of stochastic forces. Only uncorrelated noises were considered in early researching of TR [1–6]. In 1998, a noise-induced transport system with an asymmetric total noise composed of two correlated symmetric noise sources has been proposed by Li and Huang [7, 8]. Recently the noise-induced transport in a spatially symmetric periodic potential with white correlation between the additive noise and the multiplicative noise are investigated by some researchers [9–13]. They showed that the correlation noises produce

W.-Q. He · C.-J. Wang · D.-C. Mei (✉)
Department of Physics, Yunnan University, Kunming 650091, People's Republic of China
e-mail: meidch@ynu.edu.cn

W.-Q. He
Department of Physics, Honghe University, Mengzi 661100, People's Republic of China

C.-J. Wang
Nonlinear Research Institute, Baoji University of Arts and Sciences, Baoji 721007, People's Republic of China

significant impact. Noise-induced transport exits as a noise correlation effect. The direction of the current is controlled by the values of the intensity of the correlated noises (λ) and a dimensionless parameter R ($R = Q/D$, D and Q, respectively, stand for the intensity of multiplicative and additive noise). However, in their work, they assumed that noises are white correlated without the correlation time of the coupling between additive and multiplicative noise. Therefore, the noise correlation time which exists in actual physical system was excluded from their studying.

Physically, the correlation time of a real noise, though small it may be, is never strictly equal to zero. For a noise with zero correlation time, its power spectral distribution is independent on frequency. Thus the total power dissipated in all frequencies is infinite, but the actual power dissipated would be somewhat less than infinite. In other words, it appears as an idealization, only valid when the time scale for its correlation is much shorter than the time scale for the relaxation of the driven process. Afterwards it is reasonable to relax this condition and include the finite correlation time. So the way of the coupling between additive and multiplicative noise, the colored correlation with the correlation time is more close to actual physical system than the white correlation in general [14].

However, the effect of the noise cross-correlation time on the transport process is still an interesting and unexposed problem. In a periodic system, we expect the correlation time should produce significant impact. The main aim of this paper is to show the effect of the correlation time on the stationary current of Brownian particle moving in a dimension periodic potential. The structure of this paper is organized as follow. In Sect. 2 a general theory for the transport in periodic potential subject to correlation between the multiplicative noise and the additive noise, which are colored correlated in form of exponential function and the current formula is derived. In Sect. 3, a brief conclusion is given.

2 Noise-Induced Current for a General Model

Consider an overdamped Brownian particle in a periodic potential $U(x)$, $U(x+L) = U(x)$ with L being the spatial period. The stochastic dynamics is governed by the Langevin equation

$$\dot{x} = f(x) + g_1(x)\xi(t) + g_2(x)\eta(t). \tag{1}$$

In (1), $f(x) = -U'(x)$ is the deterministic force, $g_1(x)$ and $g_2(x)$ are two multiplicative functions of noise, $\xi(t)$ and $\eta(t)$ are Gaussian white noise sources with zero mean, and

$$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{2}$$

$$\langle \eta(t)\eta(t')\rangle = 2Q\delta(t-t'), \tag{3}$$

where Q and D are intensities of the noises. Here we assume

$$\langle \xi(t)\eta(t')\rangle = \langle \eta(t)\xi(t')\rangle = \frac{\lambda\sqrt{DQ}}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right] \to 2\lambda\sqrt{QD}\delta(t-t') \quad \text{as } \tau \to 0, \tag{4}$$

in which τ is the noise correlation time and λ is the intensity of the correlation between $\xi(t)$ and $\eta(t)$. We assume that (1) is a Stratonovich stochastic differential equation. Making use of the small-τ approximate approach, the approximate Fokker-Planck equation for (1) with (2–4) is obtained [15, 16]

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}f(x)P(x,t) + D\frac{\partial}{\partial x}g_1(x)\frac{\partial}{\partial x}g_1(x)P(x,t) + Q\frac{\partial}{\partial x}g_2(x)\frac{\partial}{\partial x}g_2(x)P(x,t) + \lambda\sqrt{QD}\frac{\partial}{\partial x}g_1(x)\frac{\partial}{\partial x}h_2(x)P(x,t) + \lambda\sqrt{QD}\frac{\partial}{\partial x}g_2(x)\frac{\partial}{\partial x}h_1(x)P(x,t), \tag{5}$$

where

$$h_1(x) = g_1(x)\{1 + \tau g_1(x)[f(x)/g_1(x)]'\}, \tag{6}$$

$$h_2(x) = g_2(x)\{1 + \tau g_2(x)[f(x)/g_2(x)]'\}. \tag{7}$$

Equation (5) can be rewritten in the standard form of the Fokker-Planck equation as

$$\frac{\partial}{\partial t}P(x,t) = \hat{L}_{FP}P(x,t), \tag{8}$$

$$\hat{L}_{FP} = -\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x), \tag{9}$$

where

$$A(x) = f(x) + Dg_1'(x)g_1(x) + Qg_2'(x)g_2(x) + \lambda\sqrt{QD}[g_1'(x)h_2(x) + g_2'(x)h_1(x)] \tag{10}$$

and

$$B(x) = Dg_1^2(x) + Qg_2^2(x) + \lambda\sqrt{QD}[g_1(x)h_2(x) + g_2(x)h_1(x)]. \tag{11}$$

By virtue of the generalized potential $\Phi(x)$ defined as

$$\Phi(x) = -\int_0^x \frac{A(y)}{B(y)}dy, \tag{12}$$

the general stationary probability current of Brownian particle is given by [9, 10]

$$J = [1 - e^{\Phi(x+L)-\Phi(x)}] \times \left[\int_0^L B(x)^{-1}e^{-\Phi(x)}dx \int_x^{x+L} e^{\Phi(y)}dy\right]^{-1}. \tag{13}$$

Making use of (13), we can illustrate noise-induced current in the periodic system. The periodic potential $U(x)$ is given by [9, 10]

$$U(x) = \begin{cases} 2dx/L - 2(n-1)d, & (n-1)L \le x < (2n-1)L/2, \\ -2dx/L + 2nd, & (2n-1)L/2 \le x < nL. \end{cases} \tag{14}$$

We assume the two multiplicative functions are satisfied

$$g_1(x) = \begin{cases} x - (n-1)L, & (n-1)L \le x < (2n-1)L/2, \\ x - nL, & (2n-1)L/2 \le x < nL \end{cases} \tag{15}$$

and

$$g_2(x) = 1. \tag{16}$$

therefore, $\xi(t)$ is a multiplicative noise while $\eta(t)$ is an additive noise.

It is clear that the $A(x)$ and $B(x)$ are periodic in x with period L. Therefore, we obtain the formula about the generalized potential $\Phi(x)$ as

$$\Phi(nL + \delta) = n\Phi(L) + \Phi(\delta) \tag{17}$$

for $\forall\delta \in \mathcal{R}$ and $\forall n \in \mathcal{N}$.

Now using the conclusions concerned above, we obtain the simplified form of the current from (13) as

$$J = J_1 \times J_2^{-1}, \tag{18}$$

where

$$J_1 = 1 - \exp[\Phi(L)] \tag{19}$$

and

$$J_2 = \int_0^L \int_0^L B(x)^{-1} \exp[\Phi(x+t) - \Phi(x)]dxdt. \tag{20}$$

From (20), we know that the independent variable value of $\Phi(x+t)$ covered the range from 0 to L. Taking (12) into account, the analytical piecewise expression of $A(x)$ and $B(x)$ in the range is obtained and it is as follow

$$A(x) = \begin{cases} Dx + \lambda\sqrt{QD} - 2d/L, & 0 \le x < L/2, \\ D(x-L) + \lambda\sqrt{QD} + 2d/L, & L/2 \le x < L, \end{cases} \tag{21}$$

and

$$B(x) = \begin{cases} Dx^2 + Q + 2\lambda\sqrt{QD}(x + d\tau/L), & 0 \le x < L/2, \\ D(x-L)^2 + Q + 2\lambda\sqrt{QD}(x - L - d\tau/L), & L/2 \le x < L. \end{cases} \tag{22}$$

Substituting (21, 22) into (12) and carrying out the integration, we obtain the general potential $\phi(x)$ as when $(n-1)L \le x \le (2n-1)L/2$

$$\begin{aligned} \phi(x) = {} & \frac{1}{2}\ln(Dx^2L + QL + 2\lambda\sqrt{DQ}xL + 2\lambda\sqrt{DQ}d\tau) \\ & - \frac{2d}{\sqrt{DL^2Q + 2DL\lambda\sqrt{DQ}d\tau - \lambda^2DQL^2}} \\ & \times \arctan\left(\frac{DxL + \lambda\sqrt{DQ}L}{\sqrt{DL^2Q + 2DL\lambda\sqrt{DQ}d\tau - \lambda^2DQL^2}}\right), \end{aligned} \tag{23}$$

when $(2n-1)L/2 \le x \le nL$

$$\begin{aligned} \phi(x) = {} & \frac{1}{2}\ln(Dx^2L - 2DL^2x + DL^3 + QL + 2\lambda\sqrt{DQ}xL - 2\lambda\sqrt{DQ}L^2 - 2\lambda\sqrt{DQ}d\tau) \\ & + \frac{2d}{\sqrt{DL^2Q - 2DL\lambda\sqrt{DQ}d\tau - \lambda^2DQL^2}} \end{aligned}$$

$$\times \arctan\left(\frac{DxL - DL^2 + 2\lambda\sqrt{DQ}L}{\sqrt{DL^2Q - 2DL\lambda\sqrt{DQ}d\tau - \lambda^2 DQL^2}}\right). \tag{24}$$

Substituting (19–24) into (18), the noise correlation time inducing current can be analyzed by means of numerical simulation. It is emphasized that the noise correlation time is satisfied with the condition $\tau < \frac{L}{2d|\lambda|}\sqrt{\frac{Q}{D}}$ for the diffusion coefficient $B(x)$ keeping positive for $\forall x \in \mathcal{R}$, which makes sure the physical validity of the Fokker-Planck equation (8), (9) we obtain above.

3 Conclusion

According to the expression (18) of the stationary current, the effect of the noise correlation time on the stationary current of Brownian particle in periodic potential can be analyzed by the numerical calculation. The results are plotted in Figs. 1, 2 on noise parameter D and τ, respectively. The conclusions can be drawn from these figures as follows.

(1) Under the condition that the multiplicative noise intensity D is strong, for the case of negative weak noise correlation ($\lambda < 0$), Fig. 1(a) shows the direction of current J reverses when τ is increasing, for the case of positive correlation ($\lambda > 0$), Fig. 1(b) also shows that direction of current J reverses symmetrically when τ is increasing. Thus, directed motion of the Brownian particle can be induced by the correlation time under the appropriate conditions. On the other hand, with τ increasing, J linearly changes.

(2) For the case of the correlated intensity λ is strong, the direction of the stationary current J can not be induced by the noise correlation time τ (see Fig. 2). However, the curves are not monotonous and possesses an extremum as D increases. It shows that the affect of the correlation time on current is deeply depend upon other physical parameter

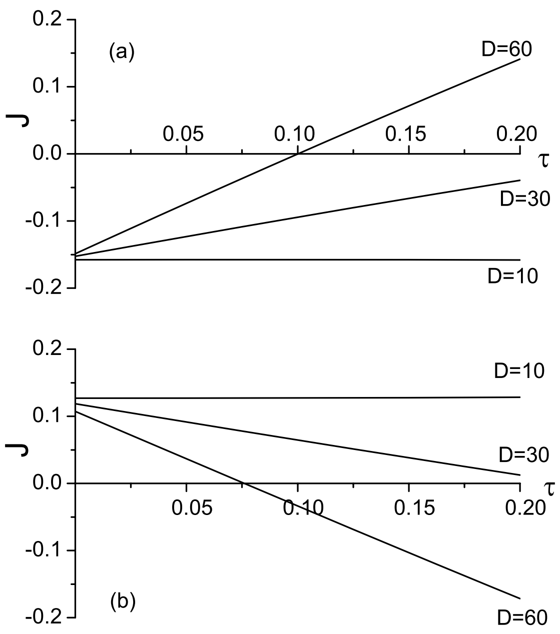

Fig. 1 Stationary current J vs τ, the curves corresponding to the value of multiplicative noise intensity D: 10, 30, 60, respectively; the other for $L = 1.0$, $d = 0.5$, $Q = 0.3$, (**a**) $\lambda = -0.1$; (**b**) $\lambda = 0.1$

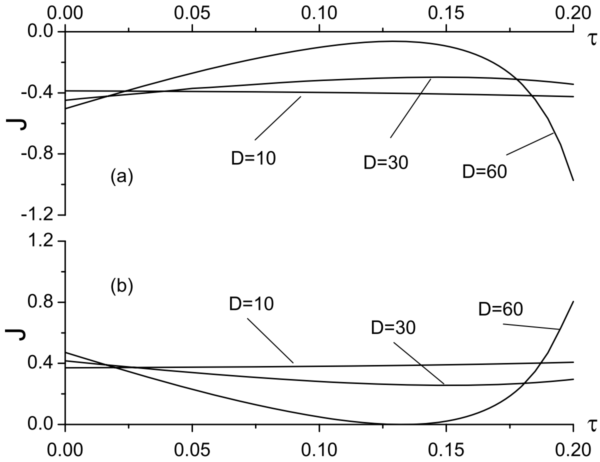

Fig. 2 Stationary current J vs τ, the curves corresponding to the value of multiplicative noise intensity D: 10, 30, 60, respectively; the other for $L = 1.0$, $d = 0.5$, $Q = 0.3$, (**a**) $\lambda = -0.3$; (**b**) $\lambda = 0.3$

such as D and λ. Therefore, the correlation time is one of the factors can influence the stationary current.

(3) Comparing Figs. 1(a) with 1(b), or Figs. 2(a) with 2(b), it is found that J is symmetric for negative correlation $-1 < \lambda < 0$ and positive correlation $0 < \lambda < 1$.

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